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Abstract

First, injective modules are one of the most popular objects in homological algebra. In most cases, base rings are commutative and Noetherian so that the testing the injectivity of a given module is an important topic. Bear's criterion for injective modules over any ring gives a big tool to classify injective modules. Every morphism from an ideal I of R should be extended to the whole ring R to be an injective module R-module. In this paper, we can show that the Baer's test can be reduced from all ideals of R to all prime ideals of R to test the injectivity of a given R-module M if the base ring R is commutative and Noetherian. Second, the Enochs' Theorem can be extended to an arbitrary sequence {ƒi} of endomomorphisms of an injective left Noetherian and if a diagram of the minimal injective resolution of RM is commutative, then the locally nilpotence of ƒ implies the locally nilpotence of other maps in the diagram.

KEYWORDS: Locally Nilpotent Endomorphism, Injective Modules, Injective Envelope

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ON INJECTIVE MODULES AND LOCALLY NILPOTENT ENDOMORPHISMS OF INJECTIVE MODULES

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ABSTRACT. First, injective modules are one of the most popular objects in homological algebra. In most cases, base rings are commutative and Noetherian so that the testing the injectivity of a given module is an important topic. Bear's criterion for injective modules over any ring gives a big tool to classify injective modules. Every morphism from an ideal I of R should be extended to the whole ring R to be an injective module R-module. In this paper, we can show that the Baer's test can be reduced from all ideals of R to all prime ideals of R to test the injectivity of a given R-module M if the base ring R is commutative and Noetherian.

Second, the Enochs' Theorem can be extended to an arbitrary sequence $\{f_i\}$ of endomomorphisms of an injective left module over a left Noetherian ring R. Moreover if the ring is left Noetherian and if a diagram of the minimal injective resolution of $_RM$ is commutative, then the locally nilpotence of f implies the locally nilpotence of other maps in the diagram.

1. A new criterion to injective modules

Proposition 1. **(Baer Criterion)** *An R-module E is injective if and only if every map $f : I \longrightarrow E$, where I is a left ideal of R can be extended to R.*

Proof. Theorem 3.20 [1]. □

If the ring is commutative and Noetherian, then we can show that the test can be weaken to all prime ideals of the ring R instead of all ideals in the ring R.

Theorem 1. *Let R be a commutative Noetherian ring and M be an R-module with a submodule A. For every $x \in M$ with $x \notin A$, there exists an $r \in R$ such that*

$$P_{rx} = \{r' \in R | r'(rx) \in A\}$$

is a prime ideal.

Proof. Consider the collection $\mathcal{C}_x$ of ideals P_{rx} with $rx \notin A$ then $P_{1\cdot x} \in \mathcal{C}_x$. Since R is Noetherian there exists a maximal element P_{rx} in $\mathcal{C}_x$ for some $r \in R$. Since $rx \notin A$, $1 \notin P_{rx}$. Suppose $st \in P_{rx}$ with $s \notin P_{rx}$. Then $srx \notin A$. So P_{srx} is in $\mathcal{C}_x$. If $y \in P_{rx}$ then $yrx \in A$ so $y(srx) = s(yrx) \in A$

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and so $P_{rx} \subset P_{srx}$. Hence by the maximality of P_{rx}, $P_{srx} = P_{rx}$. But $t(srx) = strx \in A$. So $t \in P_{srx} = P_{rx}$. Therefore P_{rx} is a prime ideal. $\square$

Theorem 2. *Let R be a commutative Noetherian ring. An R-module E is injective if and only if every map $f : I \longrightarrow E$, where I is a prime ideal in R, can be extended to R.*

Proof. Suppose E is injective, then the hypothesis is just a special case of the definition of injective modules. Suppose we have the diagram

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & A \arrow[u, "f"] \arrow[r, "i"] & B \end{tikzcd}$$

Let $\mathcal{G}$ consist of all pairs (A', g'), where $A \subset A' \subset B$ and $g' : A' \to E$ extends f. Note that $\mathcal{G} \neq \phi$, for $(A, f) \in \mathcal{G}$. Partially order $\mathcal{G}$ by saying $(A_1, g_1) \leq (A_2, g_2)$ if $A_1 \subset A_2$ and g_2 extends g_1. By Zorn's lemma, there is a maximal pair (A_0, g_0) in $\mathcal{G}$. Assume that $A_0 \neq B$. Let $x \in B - A_0$. Then there exists an $r \in R$ such that $P_{rx} = \{r' \in R | r'(rx) \in A_0\}$ is a prime ideal. Define

$$\begin{aligned} h : P_{rx} &\longrightarrow E \\ r' &\longmapsto g_0(r'rx) \end{aligned}$$

By hypothesis there exists $h' : R \to E$ extending h. Define $A_1 = A_0 + R(rx)$ and define

$$\begin{aligned} g_1 : A_1 &\to E \\ a_0 + s(rx) &\mapsto g_0 a_0 + sh'(1) \end{aligned}$$

where $s \in R$.

First, g_1 is well-defined: if $a_0 + s(rx) = a_0' + s'(rx)$ then $(s - s')(rx) = a_0' - a_0 \in A_0$ so $s - s' \in P_{rx}$. Therefore $g_0((s - s')(rx))$ and $h((s - s')(rx))$ are defined and we have

$$\begin{aligned} g_0(a_0' - a_0) &= g_0((s - s')(rx)) = h((s - s')) \\ &= h'(s - s') = (s - s')h'(1) \end{aligned}$$

Thus $g_0(a_0') - g_0(a0) = sh'(1) - s'h'(1)$ and $g_0(a_0') + s'h'(1) = g_0(a_0) + sh'(1)$. Second, g_1 extends g_0, for $g_1(a_0) = g_0(a_0)$ for all $a_0 \in A_0$. The pair (A_1, g_1) lies in $\mathcal{G}$ and is larger than the maximal pair (A_0, g_0), a contradiction. Therefore $A_0 = B$ and E is injective. This completes the proof of the theorem. $\square$

2. Locally Nilpotent Endomorphisms over Injective Modules

Definition 1. *Given any module M and $f \in End(M)$ we say f is locally nilpotent on M if for every $x \in M$, there exists $n \geq 1$ such that $f^n(x) = 0$.*

Definition 2. *An essestial extension of a module M is a module E containing M such that every nonzero submodule of E meets M (i.e., if $S \subset E$ and $S \neq 0$, then $S \bigcap M \neq 0$). We denote by $M \subset' E$.*

Lemma 1. *If E_1 and E_2 are injective envelopes of M, then any linear map $g : E_1 \longrightarrow E_2$ such that $g|_M$ is an automorphism of M is an isomorphism from E_1 to E_2.*

Proof. Since $M \subset' E_1$ is essential and $g|_M$ is injective the fact $0 = \ker(g|_M) = \ker g \bigcap M$ implies $\ker g = 0$ i.e., $g(E_1) \cong E_1$. Hence $g(E_1)$ is an injective submodule of E_2. So $E_2 = g(E_1) \oplus S$ for some S in E_2 with $g(E_1) \bigcap S = 0$. Since $M \subset g(E_1)$, we get $M \bigcap S = 0$. So the fact $M \subset' E_2$ implies $S = 0$. So $E_2 = g(E_1)$. Hence g is an isomorphism of E_1 to E_2. □

The following proposition first occured in Mathis' Thesis at University of Chicago in 1958 but only a special case. It was his Theorem 3.4 on page 520 of his article. But his result only applied if the ring R is commutative and if the map $f : E \to E$ is a multiplication by an element r of R. The more general result follows from Proposition 4.2 [4]. Remark 2 of page 200 of that paper shows how to get Mathis' result from that theorem. But the result Proposition 4.2 [4] also implies in a similar manner that if M has E as an injectice envelope and if R is left Noetherian then $f : E \to E$ such that $f(M) = 0$ is locally nilpotent on E.

Proposition 2. *If R is left Noetherian, E is an injective left R-module, and $f \in End(E)$ is such that $ker(f) \subset' E$, then f is locally nilpotent on E.*

Proof. Proposition 4.2 [4]. □

Example) If R is commutative, $r \in R$ and E is an R-module, then $E \xrightarrow{r} E (x \mapsto rx)$ is linear. Hence if E is injective and R is commutative Noetherian and if $ker(r) = K \subset' E$, then we can apply the Proposition 2 and get for any $x \in E$, $r^n x = 0$ for some $n \geq 1$.

Example) $Z(p^1) \subset' Z(p^\infty)$. But $p(Z(p^1)) = 0$. So every element in $Z(p^\infty)$ has order a power of p.

If we only assume $ker(f^2) \subset' E$ in the above, then f^2 is locally nilpotent in E and hence f itself is locally nilpotent on E. Similarly if $ker(f^n) \subset' E$ then f is locally nilpotent on E. Also note that $ker(f) \subseteq ker(f^2) \subseteq ker(f^3) \subseteq \cdots$ for any endomorphism $f : E \longrightarrow E$.

Theorem 3. *If R is left Noetherian, E is an injective left R-module, and $f \in End(E)$ is such that $\bigcup_{n=1}^{\infty} ker(f^n) \subset' E$, then f is locally nilpotent on E .*

Proof. Let $K = \cup_{n=1}^{\infty} ker(f^n) \subset' E$. Define

$$\begin{array}{rcl} \psi : E \oplus E \oplus \cdots & \longrightarrow & E \oplus E \oplus \cdots \\ (x_1, x_2, \cdots) & \longmapsto & (x_1, x_2 - f(x_1), x_3 - f(x_2), \cdots) \end{array}$$

Then easily ψ is a homomorphism. If $(x_1, x_2, \cdots)$ in $K \oplus K \oplus \cdots$, then there exist $a_i, b_i \geq 1$ such that $f^{a_i}(x_i) = 0$ and $f^{b_i}(x_{i-1}) = 0$ So $f^{m_i}(x_i - f(x_{i-1})) = 0$ for some $m_i \geq max\{a_i, b_i\}$ for each i. So ψ maps $K \oplus K \oplus \cdots$ into $K \oplus K \oplus \cdots$. And $\psi|_{K \oplus K \oplus \cdots}$ is injective since its kernel is 0. Now let $(y_1, y_2, \cdots) \in K \oplus K \oplus \cdots$. Since each $y_i \in K = \bigcup_{n=1}^{\infty} Ker(f^n)$ there exists $t_i \in \{1, 2, \cdots\}$ such that $f^{t_i}(y_i) = 0$. Given $n \in \{1, 2, \cdots\}$ choose m such that

$$m \geq max\{t_n, t_{n-1} - 1, t_{n-2} - 2, \cdots, t_1 - (n-1)\}.$$

Then

$$\begin{array}{rl} & f^m(y_n + f(y_{n-1}) + f^2(y_{n-2}) + \cdots + f^{n-1}(y_1)) \\ = & f^m(y_n) + f^{m+1}(y_{n-1}) + f^{m+2}(y_{n-2}) + \cdots + f^{m+n-1}(y_1) \\ = & 0 \end{array}$$

so for every $n \in \{1, 2, \cdots\}$

$$y_n + f(y_{n-1}) + f^2(y_{n-2}) + \cdots + f^{n-1}(y_1) \in K$$

Then

$$\begin{array}{rl} & \psi(y_1, y_2 + f(y_1), \cdots, y_n + f(y_{n-1}) + f^2(y_{n-2}) + \cdots + f^{n-1}(y_1), \cdots) \\ = & (y_1, y_2 + f(y_1) - f(y_1), y_3 + f(y_2) + f^2(y_1) - f(y_2) - f^2(y_1), \cdots) \\ = & (y_1, y_2, \cdots, y_n, \cdots). \end{array}$$

So $\psi|_{K \oplus K \oplus \cdots}$ is onto. Hence $\psi|_{K \oplus K \oplus \cdots}$ is an isomorphism on $K \oplus K \oplus \cdots$ and by the Lemma 1 ψ is an automorphism of $E \oplus E \oplus \cdots$. Let $x \in E$ and consider $(x, 0, 0, \cdots)$. Then $\psi(x_1, x_2, \cdots) = (x, 0, 0, \cdots)$ for some $(x_1, x_2, \cdots) \in E \oplus E \oplus \cdots$. Then $x_1 = x, x_2 - f(x_1) = 0, x_3 - f(x_2) = 0, \cdots, x_n - f(x_{n-1}) = 0, \cdots$. So $x_n = f(x_{n-1}) = f^2(x_{n-2}) = \cdots = f^{n-1}(x_1) = f^{n-1}(x)$. But for some n, $x_{n+1} = 0$ i.e., $f^n(x) = 0$. Therefore f is a locally nilpotent on E. □

Corollary 1. *If R is a left Noetherian ring and $f : E \longrightarrow E$ is an endomorphism of an injective left R-module such that $\cup_{n=1}^{\infty} ker(f^n) \subset' E$ then $\cup_{n=1}^{\infty} ker(f^n) = E$.*

Proof. By Theorem 3, $\cup_{n=1}^{\infty} ker(f^n) \subset' E$ implies f is locally nilpotent on E, so we easily obtain $\cup_{n=1}^{\infty} ker(f^n) = E$. □

Now we consider an arbitrary sequence $f_0, f_1, f_2, \cdots$ in $End(E)$.

Lemma 2. *Let R be a left Noetherian ring. Let E be an injective left R-module and let $f_0, f_1, f_2, \cdots$ be a sequence of elements in $End(E)$. Consider those $x \in E$ such that $f_n \circ f_{n-1} \circ \cdots f_1 \circ f_0(x) = 0$ for some n(depending on x). If K is the set of such x, then K is a submodule of E.*

Proof. Obvious. □

Theorem 4. *Let R be a left Noetherian ring, E be an injective left R-module and $f_0, f_1, f_2, \cdots$ be a sequence of elements in $End(E)$. Let $K = \{x \in E | f_n \circ f_{n-1} \circ \cdots \circ f_0(x) = 0$ for some $n \geq 1\}$. If $f_i(K) \subset K$ for all $i \geq 0$ and $K \subset' E$ then $K = E$.*

Proof. Consider the direct sum $K \oplus K \oplus \cdots$ of a countable number of Ks. Then $E \oplus E \oplus \cdots$ is easily an essential extension of $K \oplus K \oplus \cdots$. Since R is left Noetherian, $E \oplus E \oplus \cdots$ is injective, so is an injective envelope of $K \oplus K \oplus \cdots$. So define a map

$$\begin{array}{rcl} \phi : E \oplus E \oplus E \oplus \cdots & \longrightarrow & E \oplus E \oplus E \oplus \cdots \\ (x_1, x_2, x_3, \cdots) & \mapsto & (x_1, x_2 - f_0(x_1), x_3 - f_1(x_2), \cdots) \end{array}$$

Then ϕ is a homomorphism, and $\phi|_{K \oplus K \oplus \cdots}$ is an injection since $\ker \phi|_{K \oplus K \oplus \cdots} = 0$. And if $(y_1, y_2, y_3, \cdots) \in K \oplus K \oplus \cdots$, then let

$$\begin{array}{rcl} x_1 & = & y_1 \\ x_2 & = & y_2 + f_0(x_1) \\ x_3 & = & y_3 + f_1(x_2) \\ & \cdots & \end{array}$$

Then

$$\begin{array}{rcl} \phi(x_1, x_2, \cdots) & = & (x_1, x_2 - f_0(x_1), x_3 - f_1(x_2), \cdots) \\ & = & (y_1, y_2 + f_0(x_1) - f_0(x_1), y_3 + f_1(x_2) - f_1(x_2), \cdots) \\ & = & (y_1, y_2, y_3, \cdots) \end{array}$$

So ϕ is onto on $K \oplus K \oplus \cdots$. Hence by the Lemma 1, $\phi|_{K \oplus K \oplus \cdots}$ is an isomorphism of $K \oplus K \oplus \cdots$. So ϕ is an isomorphism of $E \oplus E \oplus \cdots$ and in particular ϕ is onto. Let $x \in E$ and consider $(x, 0, 0, \cdots)$. Then $\phi(x_1, x_2, x_3, \cdots) = (x, 0, 0, \cdots)$ for some $(x_1, x_2, x_3, \cdots) \in E \oplus E \oplus \cdots$. Then $x_1 = x, x_2 - f_0(x_1) = 0, x_3 - f_1(x_2) = 0$ and so on. So $x_n = f_{n-2}(x_{n-1})$ for all $n \geq 2$. But for some n, $x_{n+2} = 0$. Then

$$\begin{array}{rcl} 0 & = & x_{n+2} \\ & = & f_n(x_{n+1}) \\ & = & f_n(f_{n-1}(x_n)) \\ & = & f_n(f_{n-1}(f_{n-2}(x_{n-1}))) \\ & = & \cdots \\ & = & (f_n \cdot f_{n-1} \cdot \cdots \cdot f_0)(x) \end{array}$$

So $x \in K$ for all $x \in E$. □

Theorem 5. *If R is left Noetherian, M is a left R-module and if the diagram of minimal injective resolution of M is commutative and f is locally*

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nilpotent on M then each f^n is also locally nilpotent on $E^n(M)$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\epsilon} & E^0(M) & \xrightarrow{d_0} & E^1(M) & \longrightarrow & \cdots \\ & & \downarrow f & & \downarrow f^0 & & \downarrow f^1 & & \\ 0 & \longrightarrow & M & \xrightarrow{\epsilon} & E^0(M) & \xrightarrow{d_0} & E^1(M) & \longrightarrow & \cdots \end{array}$$

Proof. Let $x \in E^0(M)$ then since $M \subset' E^0(M)$, there exists $r \in R$ such that $rx \in M$. Since the first square is commutative, for all $y \in M$, $f(y) = f^0(y)$. The fact that f is locally nilpotent on M implies $(f^0)^n(rx) = f^n(rx) = 0$ for some $n \geq 1$. So $rx \in ker(f^0)^n$ for some n. Therefore we can say $\cup_{n=1}^{\infty} ker((f^0)^n) \subset' E^0(M)$. Hence by the Theorem 4, f^0 is locally nilpotent on $E^0(M)$.

Given $x \in E^1(M)$, there exists $r \in R$ such that $rx \in E^0(M)/M$ since the sequence is a minimal injective resolution. So $rx = y + M$ for some $y \in E^0(M)$. Then $(f^1 \circ d^0)(y) = (d^0 \circ f^0)(y)$. Since f^0 is locally nilpotent on $E^0(M)$, $(f^0)^n(y) = 0$ for some n . So $(f^0)^n(y) + M = 0 + M$. Hence

$$\begin{aligned} (f^1)^n(rx) &= (f^1)^n(y+M) \\ &= (f^1)^n(d^0(y)) \\ &= (f^1)^{n-1}(f^1 d^0(y)) \\ &= (f^1)^{n-1}(d^0 f^0(y)) \\ &= (f^1)^{n-2}(f^1 d^0 f^0(y)) \\ &= (f^1)^{n-2}(d^0 (f^0)^2(y)) \\ &= \cdots \\ &= d^0(f^0)^n(y) = 0 \end{aligned}$$

So $rx \in ker(f^1)^n$ for some n. Therefore $\cup_{n=1}^{\infty} ker(f^1)^n \subset' E^1(M)$. By the Theorem 4 f^1 is locally nilpotent on $E^1(M)$. Similarly f^i is locally nilpotent on $E^i(M)$ for each $i \geq 2$. □

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